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AN ABSTRACT MODEL OF ECONOMIC DYNAMICS FOR SECOND ORDER DISCRETE INCLUSIONS*

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Abstract. The paper deals with the abstract model of economic dynamics described by second order discrete inclusions. Then based on the concept of infimal convolution, we construct dual problems for discrete inclusions and prove the duality results. It seems that the Euler-Lagrange type inclusions are "duality relations" for both primal and dual problems. For a set-valued mapping, the graph of which is a cone, the locally adjoint mapping is calculated, and then the necessary and sufficient conditions of optimality are formulated in terms of prices. The Neumann-Gale model is investigated in detail. For the optimal trajectory, there are such prices that when choosing the intensities of the technological capacity of production at a given moment in time, the optimal trajectory corresponds to the one that provides the maximum income in the prices of the next instant time. Finally, in term of prices duality in problems with second order model with polyhedral discrete inclusions are considered.

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1 Introduction

Many abstract models of economic dynamics for discrete inclusions, macroeconomic problems are described in terms of set-valued mappings and form a component part of the modern mathematical theory of mathematical economics. The first and second-order ordinary discrete and differential inclusions, naturally arising from certain physical, economical and control problems, have attracted the attention of many researchers. Much earlier, Tarafdar (1991) and Bagh (1998) and then Urai (2010) published an important paper devoted to equilibrium in abstract economies; Bagh (1998) uses graph convergence of set -valued maps to show the existence of an equilibrium for an abstract economy without assuming the lower semicontinuity of the constraint maps. Fujimoto & Oshime (1994) and Levin (1991) provide some applications of set-valued mappings in mathematical economics. The article Freni et al. (2008) investigate an optimal control problem with mixed constraints associated with a certain linear model and establish a set of necessary and sufficient conditions for optimal trajectories, involving so-called co-state inclusion, which can be interpreted as the existence of a dual path of prices supporting the optimal path. A dynamic programming approach is also being developed, proving that the

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value function is a bilateral viscosity solution to the coupled Hamilton-Jacobi-Bellman equation. Bevia et al. (2003) present a set of axioms guaranteeing that, in exchange economies with or without indivisible goods, the set of Nash, Strong and active Walrasian Equilibria all coincide in the framework of market games. Quah & Strulovici (2013) provide another set of sufficient conditions on the primitives of the problem when both utility and discount functions depend on some one-dimensional state. The article by Medvegyev (2013) discusses an extension of the classical problem of the positivity of the growth rate in the Neumann-Gale model. It is shown that the exploitation of the labor is a necessary and sufficient condition for the positivity of the growth rate. The mathematical literature presents (Aubin & Cellina (1984); Tarafdar (1991); Bao & Mordukhovich (2014); Mordukhovich (2000) and references therein) a systematic approach to problems in economic equilibrium based on fixed-point arguments and rigorous set-theoretical (axiomatic) methods. It describes the highest-level research on the classical theme, fixed points and economic equilibria, in the theory of mathematical economics, and also presents basic results in this area, especially in the general equilibrium theory and non-co-operative game theory. In Budak et al. (2021), a nonlinear optimization model is developed to determine the price suggestions. The developed model will help to increase the profits of logistics companies in the long term.

Along with the so-called primal problems, the construction and study of the corresponding dual problems has always been relevant. Often, duality is associated with convex problems, yet it turns out that duality theory also has a fundamental impact even on the analysis of nonconvex problems. Dual Variables are so important that they have many names. Mathematicians will call them dual variables. In more general contexts, they may call them Lagrange multipliers (or just multipliers). Economists may call them dual prices, shadow prices, or implicit prices. The shadow price is also defined as the rate of change in the optimal objective function value with respect to the unit change in the availability of a resource. Thus, optimal value of the dual variable represents the per unit price (worth or marginal price) of the first resource.

From expressions of parameters of both primal and dual problems, it is clear that for the unit of measurement to be consistent, the dual variable must be expressed in terms of return (or worth) per unit of resource. This is called dual price (simplex multiplier or shadow price) of resource. In other words, optimal value of a dual variable associated with a particular primal constraint indicates the marginal change (increase, if positive or decrease, if negative) in the optimal value of the primal objective function. Duality is also used to solve a linear programming problem by the simplex method, in which the initial solution is infeasible (the technique is known as the dual simplex method). Duality theory is, in fact, closely related to game theory. In many contexts, a decision-maker does not operate in isolation, but in contend with other decisionmakers with conflicting objectives. Game theory is one approach for dealing with these "multiperson" decision problems. It views the decision-making problem as a game in which each decision-maker, or player, chooses a strategy or an action to be taken. When all players have selected a strategy, each individual player receives a payoff. Moreover, when solving linear programming problems, if some shadow price is positive, then the corresponding constraint must be satisfied with equality. In addition, if the constraint of the primary object is optional, then its corresponding shadow price must be zero. More recently, Maccheroni (2004) studies Yaari's dual theory without completeness axiom. The foundation for this type of analysis is due to Mahmudov (2005); Mahmudov & Mastaliyeva (2015); Mahmudov (2018a), Mahmudov (2018b); Mahmudov & Yusubov (2021); Mahmudov (2021a); Mahmudov (2021b); Mahmudov & Mardanov (2022) for various (ordinary and partial) optimization problems with differential inclusions, using the apparatus of infimal convolution (Rockafellar (1972); Mahmudov (2011)) and locally adjoint mappings, duality problems have been successfully constructed, and, consequently, the duality theorem is proved and duality relation is established. Therefore, in the case of weak duality, the cost of the primal problem is more than the cost of dual. In the case of strong duality, these values are equal. It is also proved that conjugate differential inclusions of the Euler-Lagrange type are simultaneously a dual relation. Conjugate duality for generalized convex optimization problems and strong Fenchel-Lagrange duality were analyzed by Dhara & Mehra (2007), Fajardol & Vidal (2018). Some methods of economics are based on the invertibility of the derivative of the Pareto frontier and cannot be applied to problems for which this frontier is not strictly concave. Cole & Kubler (2012) show how these methods can be extended to a weakly concave Pareto frontier by expanding the state space.

In addition, many authors have considered questions of existence and various qualitative problems concerning discrete and differential inclusions. The articles of Auslender (1994), Kourogenis (2003), Zhang & Li (2009) are devoted to the qualitative problems of secondorder differential inclusions, where the existence results for second-order differential inclusions are studied. Agarwal et al. (2003) and their followers considered second-order discrete inclusions in terms of set-valued mappings and proved existence theorems for feasible solutions.

In the present work, the optimality conditions for a second order discrete inclusions (DSIs) together with their duality approach were considered for the first time. In turn, for the duality problem of a discrete-approximate problem of the second order, skilfully computations of adjoint and support functions are required. Building on these results, we then treat dual results according to the dual operations of addition and infimal convolution of convex functions.

Thus, this article is devoted to one of the difficult and interesting areas - the construction of duality for an abstract model of economic dynamics and a separate study of the Neumann-Gale model with second order DSIs.

The posed problems and their dualities are new. The paper is organized in the following order:

In Section 2, the needed facts and supplementary results from the book of Mahmudov (2011) are given; Hamiltonian function H and argmaximum sets of a set-valued mapping F, Kakutani's fixed point theorem, the locally adjoint mapping (LAM), infimal convolution of proper convex functions, and the problems for second order DSIs (P_D) with endpoint constraints are formulated.

In Section 3 necessary and sufficient conditions of optimality for an abstract model of economic dynamics with second order DSIs is proved; for the optimality of the feasible solution $\{x_t\}_{t=0}^{T}$ of the primal problem, a pair of vectors $\{x_t^*\}, \{\mu_t^*\}$ satisfying the Euler-Lagrange type inclusion is required. Moreover, if a certain "nondegeneracy condition" is satisfied, that is, the standard condition of convex analysis for the presence of an interior point, then the necessary condition is at the same time sufficient for optimality.

In Section 4 the dual problem for abstract model of economic dynamics with second order DSIs (P_D) is constructed. In what follows, we prove that if β and β^* are the values of primal and dual problems, respectively, then $\beta \geq \beta^*$ for all feasible solutions. This inequality expresses the weak duality property, if for $\{x_t\}_{t=0}^{T}$ and $\{x_t^*\}, \{\mu_t^*\}$ the inequality $\beta > \beta^*$ holds. Besides, this inequality is interpreted as: the worth of resources is more than profit. Thus, so long as the total profit (return) from all activities is less than the worth of the resources, the feasible solution of both primal and dual is not optimal. The optimality (maximum profit or return) is reached only when the resources have been completely utilized. This is only possible if the worth of the resources (i.e. input) is equal to profit (i.e. output). Moreover, under the nondegeneracy condition, the existence of a solution to one of these problems implies the existence of a solution to the other problem, where $\beta = \beta^*$, and in the case where $\beta > -\infty$ the dual problem has a solution. Equality $\beta = \beta^*$ means that there is a strong duality property. It means that the EulerLagrange type adjoint inclusion at the same time is a dual relation, that is a pair of solutions of primal and dual problems satisfies this relation. It is not a coincidence that the maximal cost in the dual equals the optimal profit in the primal and that the optimal solution of the dual is the vector of shadow prices-these are fundamental relations between the primal and the dual. Finally, duality relationship between a pair of optimization problems with endpoint constraint established; it is proved that the Euler-Lagrange type adjoint DSI at the same time is a dual relation.

Section 5 is devoted to duality in problems with second order discrete model with polyhedral DSIs. The dual variable is interpreted as a price vector of a resource. In economic terminology the equality to zero of some component of this vector means the price of goods that are oversupplied must drop to zero by the law of supply and demand. This fact is what justifies interpreting the objective for the dual problem as maximizing the total implicit value of the resources consumed, rather than the resources allocated

Section 6 is devoted to the model of economic dynamics described by the second order DSI, i.e. multivalued mapping, the graph of which is a cone in the space $\mathbb{R}^{3n}$. Based on Lemma 1, the LAM is calculated and Theorem 1 is applied for the necessary and sufficient optimality conditions. Then is considered an interesting special case - the so-called Neumann-Gale model. It is proved that for the optimal trajectory, there are such prices that when the intensities of technological capacity of manufacture are selected at the instant time $(t, t+1)$, the optimal trajectory corresponds to the one that provides the maximum income in the prices of the instant time $t+2$.

2 Needed Facts and Problem Statement

Further, for the convenience of the reader, all the necessary concepts, definitions of a convex analysis can be found in the book of Mahmudov (2011) . Let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x, u\rangle$ be an inner product of elements $x, y \in \mathbb{R}^n$, and (x, y) be a pair of x, y. Assume that $G : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ into the set of subsets of $\mathbb{R}^n$. Then $G : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is convex if its gph $G = \{(x, y, z) : z \in G(x, y)\}$ is a convex subset of $\mathbb{R}^{3n}$. The set-valued mapping G is convex closed if its graph is a convex closed set in $\mathbb{R}^{3n}$. The domain of G is denoted by $\operatorname{dom} G$ and is defined as follows $\operatorname{dom} G = \{(x, y) : G(x, y) \neq \varnothing\}.G$ is convex-valued if $G(x, y)$ is a convex set for each $(x, y) \in domG$. Let us introduce the Hamiltonian function and argmaximum set for a set-valued mapping G

$$H_G(x, y, z^*) = \sup_z \{\langle z, z^*\rangle : z \in G(x, y)\}, z^* \in \mathbb{R}^n,$$
$$G_A(x, y, z^*) = \{z \in G(x, y) : \langle z, z^*\rangle = H_G(x, y, z^*)\},$$

respectively. For convex G we set $H_G(x, y, z^*) = -\infty$ if $G(x, y) = \varnothing$. As usual, $W_A(x^*)$ is a support function of the set $A \subset \mathbb{R}^n$, i.e., $W_A(x^*) = \sup_x \{\langle x, x^*\rangle : x \in A\}, x^* \in \mathbb{R}^n$.

Let int A be the interior of the set $A \subset \mathbb{R}^{3n}$ and ri A be the relative interior of the set A, i.e. the set of interior points of A with respect to its affine hull Aff A.

The convex cone $K_A(w_0), w_0 = (x_0, y_0, z_0)$ is called the cone of tangent directions at a point $w_0 \in A$ to the set A if from $\bar{w} = (\bar{x}, \bar{y}, \bar{z}) \in K_A(w_0)$ it follows that $\bar{w}$ is a tangent vector to the set A at point $w_0 \in A$, i.e., there exists such function $\eta : \mathbb{R}^1 \to \mathbb{R}^{3n}$ that $w_0 + \lambda\bar{w} + \eta(\lambda) \in A$ for sufficiently small $\lambda > 0$ and $\lambda^{-1}\eta(\lambda) \to 0$, as $\lambda \downarrow 0$.

A function φ is called a proper function if it does not assume the value $-\infty$ and is not identically equal to $+\infty$. Obviously, φ is proper if and only if $\operatorname{dom}\varphi \neq \varnothing$ and $\varphi(x, y)$ is finite for $(x, y) \in \operatorname{dom}\varphi = \{(x, y) : \varphi(x, y) < +\infty\}$.

In general, for a set-valued mapping G a set-valued mapping $G^*(\cdot, x, y, z) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ defined by

$$G^*(z^*; (x, y, z)) := \{(x^*, y^*) : (x^*, y^*, -z^*) \in K_G^*(x, y, z)\},$$

is called the LAM to a set-valued G at a point $(x, y, z) \in gphG$, where $K_G^*(x, y, z) \equiv K_{gphG}^*(x, y, z)$ is the dual to the cone of tangent directions $K_{gphG}(x, y, z)$. We provide another definition of LAM to mapping G which is more relevant for further development

$$G^*(z^*; (x, y, z)) := \{(x^*, y^*) : H_G(x_1, y_1, z^*) - H_G(x, y, z^*) \leq \langle x^*, x_1 - x\rangle + \langle y^*, y_1 - y\rangle, \forall (x_1, y_1) \in \mathbb{R}^{2n}\}, \quad (x, y, z) \in gphG, \quad z \in G_A(x, y, z^*).$$

Clearly, for the convex mapping the Hamiltonian $H(\cdot,\cdot,z^*)$ is concave and the latter and previous definitions of LAMs coincide.

Definition 1. *A function $\varphi(x,y)$ is said to be a closure if its epigraph epi $\varphi = \{(x^0,x,y) : x^0 \geq \varphi(x,y)\}$ is a closed set.*

Definition 2. *The function $\varphi^*(x^*,y^*) = \sup_{x,y}\{\langle x,x^*\rangle + \langle y,y^*\rangle - \varphi(x,y)\}$ is called the conjugate of φ. It is clear to see that the conjugate function is closed and convex.*

Let us denote

$$M_G(x^*,y^*,z^*) = \inf_{x,y,z}\{\langle x,x^*\rangle + \langle y,y^*\rangle - \langle z,z^*\rangle : (x,y,z) \in \operatorname{gph} G\},$$

that is, for every $(x,y) \in \mathbb{R}^{2n}$

$$M_G(x^*,y^*,z^*) \leq \langle x,x^*\rangle + \langle y,y^*\rangle - H_G(x,y,z^*).$$

It is clear that the function

$$M_G(x^*,y^*,z^*) = \inf_{x,y}\{\langle x,x^*\rangle + \langle y,y^*\rangle - H_G(x,y,z^*)\}$$

is a support function taken with a minus sign. Besides, it follows that for a fixed z^*

$$M_G(x^*,y^*,z^*) = -[H_G(\cdot,\cdot,z^*)]^*(x^*,y^*)$$

that is, M_G is the conjugate function for $H_G(\cdot,\cdot,z^*)$ taken with a minus sign.

Definition 3. *We recall that the operation of infimal convolution $\oplus$ of functions $g_i, i = 1,\ldots,k$ is defined as follows*

$$(g_1 \oplus \cdots \oplus g_k)(u) = \inf\left\{g_1(u^1) + \ldots + g_k\left(u^k\right) : u^1 + \ldots + u^k = u\right\}, u^i \in \mathbb{R}^n, i = 1,\ldots,k.$$

The infimal convolution $(g_1 \oplus \cdots \oplus g_k)$ is said to be exact provided the infimum above is attained for every $u \in \mathbb{R}^n$. One has $\operatorname{dom}(g_1 \oplus \cdots \oplus g_k) = \sum_{i=1}^{i=k} \operatorname{dom} g_i$. Besides for a proper convex closed function $g_i, i = 1,\ldots,k$ their infimal convolution $(g_1 \oplus \cdots \oplus g_k)$ is convex and closed (but not necessarily proper). If $g_i, i = 1,\ldots,k$ are functions not identically equal to $+\infty$, then $(g_1 \oplus \cdots \oplus g_k)^* = \sum_{i=1}^{i=k} g_i^*$. Thus, the conjugate of infimal convolution is the sum of the conjugates and this holds without any requirement on the convex functions. The operations $+$ and $\oplus$ are thus dual to each other with respect to taking conjugates.

Let us now try to describe the functioning of a certain economic system at an abstract level. Let us assume that this functioning occurs discretely in time, i.e. time t takes values $0,1,,,T$.

Definition 4. *(Kakutani's fixed point theorem) Let S be a compact, convex subset of $\mathbb{R}^n$ and let $U : S \rightrightarrows \mathbb{R}^n$ be a closed convex-valued map. Then U has a fixed point.*

Moreover, if at time $(t,t+1)$ there is a vector of resources $(x,y) \in \mathbb{R}^{2n}$, then due to the activity of production, this vector at time $t+2$ can be processed into one of the vectors $z \in F(x,y)$, where $F(x,y)$ is a convex multivalued mapping, $F(x,y) \subseteq \mathbb{R}^n$. Thus, the possible amount of (x_t,x_{t+1}) resources, over time periods, are related by the ratios $x_{t+2} \in F(x_t,x_{t+1}), t = 0,\ldots,T-2$. Let us assume that the vector of initial resources (x_0,x_1) is given. If the function $\varphi(x_{T-1},x_T)$ is interpreted as costs, then the problem posed is the problem of choosing a trajectory leading to a given set of finite resources Q with minimum total costs.

Section 3 is concerned with the following second order discrete model labelled as (P_D) :

$$\text{infimum } \varphi(x_{T-1},x_T), \tag{1}$$

$$(P_D) \qquad x_{t+2} \in F(x_t,x_{t+1}), \quad x_0 = v_0, x_1 = v_1, \tag{2}$$

$$t = 0,\ldots,T-2, \quad x_T \in Q. \tag{3}$$

A sequence $\{x_t\}_{t=0}^{T} = \{x_t : t = 0, 1, \ldots, T\}$ is called a feasible trajectory for the stated problem (1)-(3). It is required find a solution $\{\tilde{x}_t\}_{t=0}^{T}$ to a problem (P_D) for the second discrete-time problem, satisfying (2), (3) and minimizing $\varphi(x_{T-1}, x_T)$. In what follows, to this end our further strategy is as follows: first to derive necessary and sufficient conditions of optimality for problem (P_D) and then to derive duality results for them.

Definition 5. *Let us say that for the convex problem (1) - (3) the nondegeneracy condition is satisfied, if for points $x_t \in \mathbb{R}^n$, one of the following cases is fulfilled:*
(i) $(x_t, x_{t+1}, x_{t+2}) \in \operatorname{ri} \operatorname{gph} F, \quad x_T \in \operatorname{ri} Q, (x_{T-1}, x_T) \in \operatorname{ri} \operatorname{dom} \varphi, \quad t = 0, \ldots, T-2,$
(ii) $(x_t, x_{t+1}, x_{t+2}) \in$ *int* $\operatorname{gph} F, t = 0, \ldots, T-2, x_T \in \operatorname{int} Q$ *(with the possible exception of one fixed t) and φ is continuous at (x_{T-1}, x_T). It follows from the nondegeneracy condition that if $\{\tilde{x}_t\}_{t=0}^{T}$ is the optimal trajectory in the problem (1)-(3), then the cones of tangent directions $K_{\mathrm{gphF}}(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2})$ are not separable and consequently, the condition of Theorem* 3.2 *from Mahmudov (2011) is satisfied.*

3 Necessary and sufficient conditions of optimality for an abstract model of economy with second order DSI

To establish optimality, we first of all reduce the problem (P_D) at hand to a convex programming problem and use its results. Let us introduce a vector $u = (x_0, x_1, \ldots, x_T) \in \mathbb{R}^m, m = n(T+1)$ and define in the space $\mathbb{R}^{n(T+1)}$ the following convex sets

$$G_t = \{u = (x_0, \ldots, x_T) : (x_t, x_{t+1}, x_{t+2}) \in gphF\}, D_0 = \{u = (x_0, \ldots, x_N) : x_0 = v_0\},$$
$$D_1 = \{u = (x_0, \ldots, x_N) : x_1 = v_1\}, D = \{u = (x_0, \ldots, x_N) : x_T \in Q\}.$$

Now, denoting $g(u) = \varphi(x_{T-1}, x_T)$ we will reduce this problem to the problem with geometric constraints. Indeed, it can easily be seen that our basic problem (1)-(3) is equivalent to the following one

$$\text{minimize } g(u) \text{ subject to } u \in A = \left(\bigcap_{t=0}^{T-2} G_t\right) \cap D \cap D_0 \cap D_1. \tag{4}$$

In the terms of first order discrete inclusions from Mahmudov (2011) we are ready to give the necessary and sufficient conditions for the problem (1)-(3).

Theorem 1. *Let F be convex set-valued mapping, φ be proper convex function and $Q \subseteq \mathbb{R}^n$ be nonempty convex set. Besides, let $\{\tilde{x}_t\}_{t=0}^{T}$ be an optimal trajectory to the second order discrete-time problem with (P_D). Then there exist vectors $x_t^*, \mu_t^*, t = 0, \ldots, T$ and a number $\alpha \in \{0, 1\}$ simultaneously not all zero, such that the adjoint Euler-Lagrange type inclusions (i), and transversality conditions (ii) hold:*
(i) $\left(x_t^* - \mu_t^*, \mu_{t+1}^*\right) \in F^*\left(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2})\right), t = 0, 1, \ldots, T-2,$
(ii) $\left(\mu_{T-1}^* - x_{T-1}^*, \mu_T^* - x_T^*\right) \in \alpha \partial_{(x,y)} \varphi(\tilde{x}_{T-1}, \tilde{x}_T), \mu_T^* \in K_Q^*(\tilde{x}_T).$
In addition, if the nondegeneracy condition is satisfied, then $\alpha = 1$ and these conditions are sufficient for optimality of the trajectory of $\{\tilde{x}_t\}_{t=0}^{T}$.

Proof. Under the nondegeneracy condition, the necessary and sufficient conditions of problem (4) are reduced to the condition $\partial_u g(\tilde{u}) \cap K_A^*(\tilde{u}) \neq \varnothing$ and its interpretation, where $\partial_u g(\tilde{u})$ is a subdifferential of g at a point $\tilde{u}$. □

4 Duality for an abstract model of economy with second-order DSI

This section is devoted to the construction of the dual problem and the establishment of the duality relationship. For construction of duality we need the following auxiliary result.

We call the following problem, labelled as (P_D^*), the dual problem to the problem with second order DSIs (P_D)

$$(P_D^*) \quad \sup_{\substack{x_t^*,\mu_t^*,\\ t=0,.,T}} \left\{ -\varphi^*\left(\mu_{T-1}^* - x_{T-1}^*, \mu_T^* - x_T^*\right) + \sum_{t=0}^{T-2} M_F\left(x_t^* - \mu_t^*, \mu_{t+1}^*, x_{t+2}^*\right) + \langle v_0, \mu_0^* - x_0^* \rangle \right.$$

$$\left. - \langle v_1, x_1^* \rangle - W_Q\left(-\mu_T^*\right) \right\},$$

where W_Q is a support function of the set Q, respectively.

Theorem 2. *If β and β^* are the optimal values of the optimization problem for second order DSIs (P_D) and its dual problem (P_D^*), respectively, then $\beta \geq \beta^*$ for all feasible solutions of primal (P_D) and dual (P_D^*) problems. Besides, if the nondegeneracy condition is satisfied, then, the existence of a solution to one of these problems implies the existence of a solution to another, where $\beta = \beta^*$ and in the case $\beta > -\infty$ the dual problem (P_D^*) has a solution.*

Proof. It is known from convex analysis that the operations of addition and infimal convolution of convex functions are dual to each other and for a problem (4) the dual problem is $sup\left\{-g^*\left(u^*\right) - \delta_A^*\left(-u^*\right)\right\}$, where $\delta_A(\cdot)$ is the indicator function of A, i.e., $\delta_A(u) = 0, u \in A$ and $\delta_A(u) = +\infty, u \notin A$. □

Theorem 3. *Suppose that $\{\tilde{x}_t\}_{t=0}^T$ is an optimal solution to primal problem (P_D) and that the nondegeneracy condition is satisfied. Besides, suppose that φ is proper convex closed function, Q and F are convex closed set and set-valued mapping, respectively. Then the family of vectors $\{x_t^*, \mu_t^*\}_{t=0}^T$ is an optimal solution to the dual problem (P_D^*) if and only if the adjoint Euler-Lagrange type inclusions (i), and transversality conditions (ii) of Theorem 1 are satisfied.*

5 Model of mathematical economics with second order polyhedral DSIs

In this section we establish the dual problem (PL^*) to the problem with the following second order polyhedral differential inclusion:

$$\begin{aligned} (PL) \qquad & \text{infimum } \varphi\left(x_{T-1}, x_T\right) \\ & x_{t+2} \in F\left(x_t, x_{t+1}\right), \quad x_0 = v_0, x_1 = v_1, \\ & t = 0, \dots, T-2, \quad x_T \in Q, Q = \{x : Dx \leq 0\}, \\ & F(x,y) = \{z : Ax + By - Cz \leq d\}, \end{aligned} \tag{5}$$

where F is a second order polyhedral set-valued mapping, A, B, C, D are $s \times n$ dimensional matrices, d is a s-dimensional column-vector, $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^1$ is a convex function, v_0, v_1 are fixed vectors. We label this problem by (PL). Then with respect to the dual problem (P_D^*) first we should calculate $M_F\left(x^*, y^*, z^*\right)$:

$$M_F\left(x^*, y^*, z^*\right) = \inf\left\{\langle x, x^* \rangle + \langle y, y^* \rangle - \langle z, z^* \rangle : (x,y,z) \in \operatorname{gph} F\right\}.$$

In fact, denoting $w = (x,y,z) \in \mathbb{R}^{3n}, w^* = (x^*, y^*, -z^*) \in \mathbb{R}^{3n}$ we have a linear programming problem

$$\inf\left\{\langle w, w^* \rangle : Sw \leq d\right\}, \tag{6}$$

where $S = [A \vdots B \vdots -C]$ is $s \times 3n$ block matrix. Then according to the linear programming theory if $\tilde{w} = (\tilde{x}, \tilde{y}, \tilde{z})$ is a solution of (6), then there exists s-dimensional vector $\lambda \geq 0$ such that

$$w^* = -S^*\lambda, \langle A\tilde{x} + B\tilde{y} - C\tilde{z} - d, \lambda \rangle = 0.$$

On the contrary, if these conditions are satisfied, then $\tilde{w} = (\tilde{x}, \tilde{y}, \tilde{z})$ is a solution of the problem (6). Hence, $w^* = -S^*\lambda$ means that $x^* = -A^*\lambda, y^* = -B^*\lambda, z^* = -C^*\lambda, \lambda \geq 0$. Thus, we find that

$$\begin{aligned} M_F(x^*, y^*, z^*) &= \langle \tilde{x}, -A^*\lambda \rangle + \langle \tilde{y}, -B^*\lambda \rangle - \langle \tilde{z}, -C^*\lambda \rangle \\ &= -\langle A\tilde{x}, \lambda \rangle - \langle B\tilde{y}, \lambda \rangle + \langle C\tilde{z}, \lambda \rangle = -\langle d, \lambda \rangle. \end{aligned} \tag{7}$$

On the other hand, from the form of $M_F\left(x_t^* - \mu_t^*, \mu_{t+1}^*, x_{t+2}^*\right)$ by Theorem 1 we derive that

$$x_t^* - \mu_t^* = -A^*\lambda_{t+2}, \mu_{t+1}^* = -B^*\lambda_{t+2}, x_{t+2}^* = -C^*\lambda_{t+2}, \lambda_t \geq 0. \tag{8}$$

Further, since the set Q is a cone it follows that

$$W_Q(x^*) = \begin{cases} 0, & \text{if } x^* \in Q^*, \\ +\infty & \text{if } x^* \notin Q^* \end{cases}$$

i.e., the support function of a cone is the indicator function of its dual cone. Then it is easy to calculate that

$$Q^* = \{x^* : x^* = -D^*\gamma, \gamma \geq 0\}, \text{ i.e., } \mu_T^* = -D^*\gamma. \tag{9}$$

Therefore, considering (7)-(9) and Theorem 1 we have the following dual problem, labelled (PL^*) :

$$\sup_{\substack{\lambda_t \geq 0, \gamma \geq 0, \\ t=0,\ldots,T}} \left\{ -\varphi^*\left(C^*\lambda_{T-1} - B^*\lambda_{T-1}, C^*\lambda_T - D^*\gamma\right) - \sum_{t=0}^{T-2} \langle d, \lambda_{t+2} \rangle + \langle v_0, A^*\lambda_2 \rangle + \langle v_1, C^*\lambda_1 \rangle \right.$$
$$\left. - W_Q(D^*\gamma) \right\}.$$

Now, before formulation of duality theorem we should proof sufficient condition of optimality for a problem (PL).

Theorem 4. *Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^1$ be continuous proper convex function and F be a polyhedral set-valued mapping given in problem (PL). Then for the optimality of the trajectory $\tilde{x}(\cdot)$ in problem* (PL) *with second order polyhedral differential inclusions, it is sufficient that there exists a function $\lambda_t \geq 0, t = 0, .., T$ and vector $\gamma \geq 0$ satisfying a.e. the following second order Euler-Lagrange type polyhedral differential inclusion and transversality condition at the endpoint $t = T$:*

(1) $A^*\lambda_{t+2} + B^*\lambda_{t+1} - C^*\lambda_t = 0, \quad , \lambda_t \geq 0, t = 0, .., T-2,$
$\langle A\tilde{x}_t + B\tilde{x}_{t+1} - C\tilde{x}_{t+2} - d, \lambda_t \rangle = 0, t = 0, \ldots, T-2,$

(2) $(C^*\lambda_{T-1} - B^*\lambda_{T-1}, C^*\lambda_T - D^*\gamma) \in \alpha\partial_{(x,y)}\varphi(\tilde{x}_{T-1}, \tilde{x}_T).$

Proof. In fact, condition (1) of the theorem immediately follows from formula (8). Further, by the conditions $(a), (b)$ of Theorem 1 we have

$$\left(\mu_{T-1}^* - x_{T-1}^*, \mu_T^* - x_T^*\right) \in \alpha\partial_{(x,y)}\varphi(\tilde{x}_{T-1}, \tilde{x}_T). \tag{10}$$

Hence, by (8) and (9) from (10) we derive that

$$(C^*\lambda_{T-1} - B^*\lambda_T, C^*\lambda_T - D^*\gamma) \in \alpha\partial_{(x,y)}\varphi(\tilde{x}_{T-1}, \tilde{x}_T).$$

The proof of theorem is completed. □

Then, as a result of Theorems 2 and 3 for a problem (PL) we have the following duality theorem.

Theorem 5. *Let the conditions of Theorem 1 be satisfied and $\{\tilde{x}_t\}_{t=0}^{T}$ be an optimal solution of the primal problem (PL). Then $\left\{\tilde{\lambda}_t\right\}_{t=0}^{T}, \tilde{\lambda}_t \geq 0, t = 0,..,T$ is an optimal solution of the dual problem (PL^*) if and only if the sufficient optimality conditions of Theorem 4 are satisfied. In addition, the optimal values in the primal* (PL) *and dual (PL^*) problems are equal.*

Remark 1. *Obviously, $\lambda_t \geq 0, t = 0,..,T$ can be interpreted as a price of a resource. Wherein, if $\lambda_t^i = 0 (i = 1, \ldots, n)$ for some i whenever the supply of this resource is not exhausted by the activities. In economic terminology, such a resource is a "free good"; the price of goods that are oversupplied must drop to zero by the law of supply and demand. This fact is what justifies interpreting the objective for the dual problem as maximizing the total implicit value of the resources consumed, rather than the resources allocated. Strong duality means the solution to these matches centralized if $\tilde{\lambda}_t$ is optimal multiplier. Strong duality guaranteed by convexity.*

6 Neumann-Gale Model for second order DSIs

In this section we consider an abstract economic model, graph of which is a cone in $\mathbb{R}^{3n}$ and then the famous Neumann-Gale model. Let $K \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be a cone, gph $F = K$, where

$$F(x, y) = \{z : (x, y, z) \in K\}.$$

In models of economic dynamics, a situation corresponding to this example is most often considered, and the model itself is called the John von Neumann-David Gale model (Neumann-Gale). The following lemma is very important for calculation of LAM in Neumann-Gale model.

Lemma 1. *Let $K \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be a cone, gph $F = K$, where $F(x, y) = \{z : (x, y, z) \in K\}$. Then at a point $(x_0, y_0, z_0) \in K$ the LAM $F^*(z^*; (x_0, y_0, z_0))$ is calculated by the formula*

$$F^*(z^*; (x_0, y_0, z_0)) := \{(x^*, y^*) : (x^*, y^*, -z^*) \in K^*, \langle x_0, x^* \rangle + \langle y_0, y^* \rangle = \langle z_0, z^* \rangle\}.$$

Proof. Let $(x_0, y_0, z_0) \in K$. Since $K_F(x_0, y_0, z_0) \equiv K_{\text{gph}F}(x_0, y_0, z_0) = \text{cone}\,(K - (x_0, y_0, z_0))$, then according to the definition $(x^*, y^*, -z^*) \in K_F^*(x_0, y_0, z_0)$ if and only if

$$\langle x - x_0, x^* \rangle + \langle y - y_0, y^* \rangle - \langle z - z_0, z^* \rangle \geq 0$$

for all $(x, y, z) \in K$. Rewriting the last inequality in the form

$$\langle x, x^* \rangle + \langle y, y^* \rangle - \langle z, z^* \rangle \geq \langle x_0, x^* \rangle + \langle y_0, y^* \rangle - \langle z_0, z^* \rangle$$

by Lemma 1.18 from Mahmudov (2011) we have $(x^*, y^*, -z^*) \in K^*$ and the lower bound of the left side of the latter inequality on $(x, y, z) \in K$ is equal to zero. Hence,

$$0 \geq \langle x_0, x^* \rangle + \langle y_0, y^* \rangle - \langle z_0, z^* \rangle. \tag{11}$$

On the other hand, since $(x_0, y_0, z_0) \in K$ and $(x^*, y^*, -z^*) \in K^*$, then

$$0 \leq \langle x_0, x^* \rangle + \langle y_0, y^* \rangle - \langle z_0, z^* \rangle. \tag{12}$$

Finally, from the inequalities (11),(12) we get, $\langle x_0, x^* \rangle + \langle y_0, y^* \rangle = \langle z_0, z^* \rangle$. Therefore,

$$F^*(z^*; (x_0, y_0, z_0)) := \{(x^*, y^*) : (x^*, y^*, -z^*) \in K^*, \langle x_0, x^* \rangle + \langle y_0, y^* \rangle = \langle z_0, z^* \rangle\}.$$

□

Let now take $Q = \mathbb{R}^n$ in problem (P_D). Then, obviously, $K_Q^*(x) = \{0\}$. Then considering into account the formula $K_Q^*(x) = \{0\}$ and Lemma 1 the conditions of Theorem 1 we can summarize as following theorem.

Theorem 6. *Let* $\operatorname{gph} F(\cdot,t) = K \subseteq \mathbb{R}^{3n}$ *and* $Q = \mathbb{R}^n$. *Besides, let* $\{\tilde{x}\}_{t=0}^{T}$ *be an optimal trajectory to the second order discrete-time problem with* (P_D). *Then there exist vectors* x_t^*, μ_t^*, $t = 0,\ldots,T$ *and a number* $\alpha \in \{0,1\}$ *simultaneously not all zero, such that the adjoint EulerLagrange type inclusions (i), and transversality condition (ii), hold:*
(i) $\left(x_t^* - \mu_t^*, \mu_{t+1}^*, -x_{t+2}^*\right) \in K^*; \langle \tilde{x}_t, x_t^* - \mu_t^* \rangle + \left\langle \tilde{x}_{t+1}, \mu_{t+1}^* \right\rangle = \left\langle \tilde{x}_{t+2}, x_{t+2}^* \right\rangle$,
(ii) $\left(\mu_{T-1}^* - x_{T-1}^*, \mu_T^* - x_T^*\right) \in \alpha \partial_{(x,y)} \varphi\left(\tilde{x}_{T-1}, \tilde{x}_T\right), t = 0,1,\ldots,T-2.$
Moreover, if the nondegeneracy condition is satisfied, then $\alpha = 1$ *and these conditions are sufficient for optimality of the trajectory of* $\{\tilde{x}_t\}_{t=0}^{T}$.

Note that with those interpretations in mind, dual optimal solutions have been termed "shadow price vectors" and "equilibrium price vectors". Previously, it was known that an $\partial_{(x,y)} \varphi\left(\tilde{x}_{T-1}, \tilde{x}_T\right)$ has an economic interpretation as the set of "equilibrium price" vectors for problem.

Consider now a special case - the so-called Neumann-Gale model. Suppose there are m technological capacity to manufacture output with unit commodity intensity leads manufacture of a_j commodity $a_j \in \mathbb{R}^n$. Thus, the number of different manufactured goods is n and under the unit commodity intensity utilization of j-th technological capacity of manufacture of i-th commodity is produced amount of $a_j^i, i = 1,\ldots,n$ goods. Naturally, we let $a_j^i \geq 0$. Here under the unit commodity intensity employment is emitted $b_j \in \mathbb{R}^n$ and $c_j \in \mathbb{R}^n$ commodities. Now, if at the given instant time there is x, y outputs, correspondingly then intensity λ^j of each manufacture capacity, obviously must satisfy the inequality

$$x \geq \sum_{i=1}^{m} b_j \lambda^j, \quad y \geq \sum_{i=1}^{m} c_j \lambda^j, b_j \geq 0, c_j \geq 0, \lambda^j \geq 0.$$

In this case, a vector z of goods will be released that satisfies the equation $z = \sum_{i=1}^{i=m} a_j \lambda^j$. Finally, taking the matrices A, B and C with columns a_j, b_j and c_j, respectively, we define a set-valued mapping $F(x,y)$, the graph of which is a polyhedral cone:

$$\begin{aligned} K &= \{(x,y,z) : x \geq B\lambda, y \geq C\lambda, z = A\lambda, \lambda \geq 0, \lambda \in \mathbb{R}^m\}, \\ F(x,y) &= \{z = A\lambda : x \geq B\lambda, y \geq C\lambda, \lambda \geq 0, \lambda \in \mathbb{R}^m\}, \end{aligned} \tag{13}$$

where λ is a vector with components $\lambda^j, j = 1,\ldots,m$. Our interpretation implies the inequalities $A \geq 0, B \geq 0$ and, i.e., the matrices A, B and $C \geq 0$ are non-negative.

As is well known, a fixed-point theorem shows the existence of equilibrium points of an abstract economy. Of course, for such a mapping, it makes sense to talk about fixed points with respect to the variables x or y separately, that is, for $F(x, y_0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $F(x_0, y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. It is clear that, by the definition of fixed points, $x \in F(x, y_0)$ and $y \in F(x_0, y)$, respectively, where x_0, y_0 are arbitrary fixed points. More specifically, it is easy to check that for the presence of a Kakutani's fixed point theorem, the inequalities $A \geq B$ and $A \geq C$, must hold, respectively.

Note that in general, Fixed-point theorems (Kakutani's, Brouwer's etc.) are a major tool in both functional analysis and mathematical economics and are used to prove the existence of solutions to differential equations and the existence of Nash equilibria among other things.
Since for $(x,y,z) \in K$

$$x = B\lambda + u, y = C\lambda + v, \lambda \geq 0, u, v \geq 0, u, v \in \mathbb{R}^n,$$

then it is easy to calculate the dual cone K^*. Indeed, $(x^*, y^*, z^*) \in K^*$ if and only if

$$\langle x, x^* \rangle + \langle y, y^* \rangle + \langle z, z^* \rangle \geq 0, (x,y,z) \in K,$$

i.e.

$$\langle B\lambda + u, x^* \rangle + \langle C\lambda + v, y^* \rangle + \langle A\lambda, z^* \rangle \geq 0 \text{ for all } \lambda \geq 0, u, v \geq 0.$$

Rewriting the last inequality in the form

$$\langle u, x^* \rangle + \langle v, y^* \rangle + \langle \lambda, B^*x^* + C^*y^* + A^*z^* \rangle \geq 0, u \geq 0, v \geq 0, \lambda \geq 0,$$

where $()^*$ stands for the matrix transposition operation. Then from the latter inequality we obtain in an obvious way the inequalities

$$x^* \geq 0, y^* \geq 0, B^*x^* + C^*y^* + A^*z^* \geq 0.$$

Therefore,

$$K^* = \{(x^*, y^*, z^*) : B^*x^* + C^*y^* + A^*z^* \geq 0, x^* \geq 0, y^* \geq 0\}. \tag{14}$$

Let now, at the initial instant time, there is a vector of goods (v_0, v_1).Then, by the time T, can be produced x_T goods, where x_T is obtained from the chain of inclusions

$$x_{t+2} \in F(x_t, x_{t+1}), x_0 = v_0, x_1 = v_1, t = 0, \ldots, T-2. \tag{15}$$

If at time T and $T-1$ the prices of goods $p^{i^*}, q^{i^*}, i = 1, \ldots, n$ are given, then the total cost of goods at time T and T-1 will be determined by the following formula

$$\sum_{i=1}^{n} p^{i^*} x_T^i + \sum_{i=1}^{n} q^{i^*} x_{T-1}^i = \langle x_T, p^* \rangle + \langle x_{T-1}, q^* \rangle,$$

where p^*, q^* are prices vectors with components $p^{i^*}, q^{i^*}, i = 1, \ldots, n$. Now we can consider the problem of choosing such a way of functioning of the economy, that is, such a trajectory that satisfies the relations (15) in order to maximize $\langle x_T, p^* \rangle + \langle x_{T-1}, q^* \rangle$. The latter is equivalent to minimizing the function

$$\varphi(x_{T-1}, x_T) = -\langle x_T, p^* \rangle - \langle x_{T-1}, q^* \rangle.$$

We arrive at the problem considered above with a special cone K and a function $\varphi(x_{T-1}, x_T)$, and the conditions of Theorem 1 can be used to characterize the optimal trajectory. Then, taking into account (14), these relations can be written in the following form

$$x_t^* \geq \mu_t^*, \mu_{t+1}^* \geq 0, B^*(x_t^* - \mu_t^*) + C^*\mu_{t+1}^* - A^*x_{t+2}^* \geq 0, \tag{16}$$

$$\langle \tilde{x}_t, x_t^* - \mu_t^* \rangle + \langle \tilde{x}_{t+1}, \mu_{t+1}^* \rangle = \langle \tilde{x}_{t+2}, x_{t+2}^* \rangle, t = 0, 1, \ldots, T-2, \tag{17}$$

$$\left(\mu_{T-1}^* - x_{T-1}^*, -x_T^*\right) \in \alpha \partial_{(x,y)} \varphi(\tilde{x}_{T-1}, \tilde{x}_T), \alpha \in \{0, 1\}. \tag{18}$$

But in (18) $\varphi(\tilde{x}_{T-1}, \tilde{x}_T) = -\langle \tilde{x}_T, p^* \rangle - \langle \tilde{x}_{T-1}, q^* \rangle$ and therefore it is easy to see that $\partial_{(x,y)} \varphi(\tilde{x}_{T-1}, \tilde{x}_T) = \{-q^*\} \times \{-p^*\}$ and consequently,

$$x_{T-1}^* - \mu_{T-1}^* = \alpha q^*, x_T^* = \alpha p^*, \alpha \in \{0, 1\}. \tag{19}$$

Note now that if the set-valued mapping F and the set Q in problem (P_D) are polyhedrally, then in its proof we can use Lemma 1.22 from Mahmudov (2011), from which it follows that in the statement of Theorem 1 we can put $\alpha = 1$. Therefore, $\alpha = 1$ in the formula (18), and hence in the formula (19).

We now put $p_t^* = x_t^*, q_t^* = \mu_t^*, t = 0, \ldots, T$, and write the relations (11)-(17) and (19) in the following form

$$\begin{gathered} p_t^* \geq q_t^*, q_{t+1}^* \geq 0, B^*(p_t^* - q_t^*) + C^*q_{t+1}^* - A^*p_{t+2}^* \geq 0, \\ \langle \tilde{x}_t, p_t^* - q_t^* \rangle + \langle \tilde{x}_{t+1}, q_{t+1}^* \rangle = \langle \tilde{x}_{t+2}, p_{t+2}^* \rangle, t = 0, 1, \ldots, T-2, \\ p_{T-1}^* - q_{T-1}^* = q^*, p_T^* = p^*. \end{gathered} \tag{20}$$

The carried out formal transformations lead to the following conclusion; the trajectory $\{\tilde{x}\}_{t=0}^T$ of the Neumann-Gale model, satisfying the relations (15), if and only if it maximizes the function of the final income, when there are vectors p_t^*, q_t^* such that relations (20) are satisfied. Consider a possible economic interpretation of the result. Let us call the vectors p_t^*, q_t^* prices at the instant time t. This makes sense, since $p_t^* \geq 0, q_t^* \geq 0$.

Let z be an arbitrary vector from $F(\tilde{x}_t, \tilde{x}_{t+1})$. It means that

$$z = A\lambda, \quad \tilde{x}_t \geq B\lambda, \quad \tilde{x}_{t+1} \geq C\lambda, \quad \lambda \geq 0.$$

Based on these relations and formulas (20), it is easy to verify the validity of the following chain of equalities and inequalities:

$$\begin{aligned} \langle z, p_{t+2}^* \rangle &= \langle A\lambda, p_{t+2}^* \rangle = \langle \lambda, A^* p_{t+2}^* \rangle + \langle \lambda, B^* (p_t^* - q_t^*) + C^* q_{t+1}^* \rangle \\ &= \langle B\lambda, p_t^* - q_t^* \rangle + \langle C\lambda, q_{t+1}^* \rangle \leq \langle \tilde{x}_t, p_t^* - q_t^* \rangle + \langle \tilde{x}_{t+1}, q_{t+1}^* \rangle . \end{aligned} \tag{21}$$

But since

$$\langle \tilde{x}_t, p_t^* - q_t^* \rangle + \langle \tilde{x}_{t+1}, q_{t+1}^* \rangle = \langle \tilde{x}_{t+2}, p_{t+2}^* \rangle ,$$

then from (21) we have

$$\langle z, p_{t+2}^* \rangle \leq \langle \tilde{x}_{t+2}, p_{t+2}^* \rangle , \quad z \in F(\tilde{x}_t, \tilde{x}_{t+1}) ,$$

which means that

$$\langle \tilde{x}_{t+2}, p_{t+2}^* \rangle = \max_z \left\{ \langle z, p_{t+2}^* \rangle : z \in F(\tilde{x}_t, \tilde{x}_{t+1}) \right\} .$$

An important conclusion follows from the obtained relation; for the optimal trajectory, there are such prices that when the intensities of technological capacity of manufacture are selected at the instant time $(t, t+1)$, the optimal trajectory corresponds to the one that provides the maximum income in the prices of the instant time $t+2$.

7 Conclusion

The paper under the "nondegeneracy" condition deals with the abstract model of economics dynamics with second order DSIs, which are often used to describe various processes in science and economics; first are derived necessary and sufficient optimality conditions in the form of Euler-Lagrange type inclusions and transversality conditions. Then we treat dual results according to the dual operations of addition and infimal convolution of convex functions. It appears that the Euler-Lagrange type inclusions are duality relations for both primal and dual problems. As an open problem for further investigations, we mention the study of NeumannGale model and duality theory for an arbitrary higher-order DSIs. On the other hand, the generalization of the considered models to the case with differential inclusions is of considerable interest. Consequently, a rather complicated Mayer problem arises with the simultaneous determination of conjugacy of φ and the appearance of higher orders. Thus, we can conclude that the proposed method is reliable for solving various duality problems with discrete inclusions of a higher order and for studying the Neumann-Gale model.

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